Research Note
Andreas Steiner Consulting GmbH
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Sharpe Ratio Contribution and Attribution Analysis

"Humans best process information when relationships are linear." – William Sharpe

Introduction

Relatively unnoticed, major advances have been made over the last eight years in the contribution and attribution analysis of risk and risk-adjusted performance[1].

This is significant since classical performance attribution approaches, e.g. Brinson decompositions, ignore risk aspects altogether. To some degree, this is due to a technical reason: both contribution and attribution analysis are inherently linear; the goal is to derive independent effects that sum across portfolio constituents as well as attributes. Risk, in contrast, is non-linear. For example, it is well known that portfolio volatility does not equal the sum of asset volatilities due to diversification effects.

At the same time, academic research produced an impressive battery of sophisticated risk-adjusted performance measures. These measures are typically computed at portfolio level for the purpose of ranking investment portfolios. In this context, there is little to no need for contribution and attribution analysis. Quite often, there is simply no information on portfolio constituent available to analysts (e.g. fund-of-funds, hedge funds).

Sharpe Ratio contribution analysis gives an answer to the question "which portfolio constituents are driving my Sharpe Ratio". Sharpe Ratio attribution analysis tries to explain differences in Sharpe Ratios in terms of differences in risk factors that were subject to conscious management decisions. What follows is a proposition of a method to calculate contributions to the Sharpe Ratio and a framework to analyze Sharpe Ratio differences.

Sharpe Ratio Contributions

Contributions to the Sharpe Ratio can be derived by simply applying a procedure known from Information Ratio decomposition. This is possible because the Sharpe Ratio is nothing else than an Information Ratio relative to a rather special benchmark: the riskfree asset, i.e.

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[1] See for example Philippe Bertrand, "Risk-adjusted performance attribution and portfolio optimizations under tracking-error constraints", Journal of Asset Management, 2009. The topic is related to the notion of "risk budgeting", see for example William Sharpe, "Budgeting and Monitoring Pension Fund Risk", Financial Analysts Journal, 2002

an asset with zero volatility. As most readers are probably not familiar yet with the information ratio decomposition, we present a detailed derivation.

Starting with the definition of the Sharpe Ratio as the quotient of portfolio excess returns R_p and portfolio volatility σ_p…

$$S_P = \frac{R_P - R_f}{\sigma_P} = \frac{r_P}{\sigma_P}$$

…r_p can be decomposed in a linear fashion into asset contributions, calculated as asset weights w_i times asset excess returns r_i…

$$S_P = \frac{\sum_{i=1}^{n} w_i \cdot r_i}{\sigma_P} = \sum_{i=1}^{n} \frac{w_i \cdot r_i}{\sigma_P}$$

The challenge is to decompose portfolio volatility. Since volatility is a so-called linear homogenous function, the weighted marginal contributions to volatility sum up to portfolio volatility…

$$\sigma_P = \sum_{i=1}^{n} w_i \cdot \frac{\partial \sigma_P}{\partial w_i}$$

It is also rather well known that the marginal contribution of an asset to portfolio volatility is its covariance $\sigma_{i,p}$ with the overall portfolio. From the definition of covariance, it follows that the marginal contribution of an asset to portfolio risk can be expressed as…

$$\frac{\partial \sigma_P}{\partial w_i} = \frac{\sigma_{i,P}}{\sigma_P} = \rho_{i,P} \cdot \sigma_i$$

This can be summarized to the following portfolio volatility decomposition…

$$\sigma_P = \sum_{i=1}^{n} w_i \cdot \rho_{i,P} \cdot \sigma_i$$

The above expression can be interpreted as follows: An asset's contribution to portfolio risk consists of three components…

1. Exposure: the weight of the asset in the portfolio
2. Asset risk: the volatility of the asset
3. Diversification: the diversification of an asset with the overall portfolio

Note that these three components can also be interpreted in the context of the classical performance attribution terminology…

1. Allocation: asset weight
2. Selection: asset volatility
3. Interaction: asset correlation with portfolio

The next step may seem odd at first; we multiply and divide by $w_i \cdot \rho_{i,P} \cdot \sigma_i / \sigma_P$ in the Sharpe Ratio definition…

$$S_P = \sum_{i=1}^{n} \frac{w_i \cdot r_i}{\sigma_P} \cdot \frac{w_i \cdot \rho_{i,P} \cdot \sigma_i}{\sigma_P} \cdot \frac{\sigma_P}{w_i \cdot \rho_{i,P} \cdot \sigma_i}$$

Now we regroup and finally arrive at…

$$\boxed{S_P = \sum_{i=1}^{n} \frac{w_i \cdot \rho_{i,P} \cdot \sigma_i}{\sigma_P} \cdot \frac{1}{\rho_{i,P}} \cdot \frac{r_i}{\sigma_i}}$$

This expression is an additive decomposition of the Sharpe Ratio, in which an asset's contribution consists of three components…

1. Risk weights $w_i \cdot \rho_{i,P} \cdot \sigma_i / \sigma_P$. These expressions are weights because they sum to unity: $\sum_{i=1}^{n} w_i \cdot \rho_{i,P} \cdot \sigma_i / \sigma_P = 1$. In fact, the risk weights are nothing else than the percentage contribution of an asset to portfolio volatility.
2. Diversification effects $1/\rho_{i,P}$, reflecting the interaction of the assets when added to the portfolio.
3. Asset Sharpe Ratios r_i / σ_i

The product of the second and third elements $\frac{1}{\rho_{i,P}} \cdot \frac{r_i}{\sigma_i}$ is sometimes called the "component Sharpe Ratio".[2]

We now apply the above decomposition to a portfolio consisting of three assets with the following characteristics…

	Assets		
	I	**II**	**III**
Portfolio Weights	**30.00%**	**40.00%**	**30.00%**
Expected Return	0.15%	0.87%	2.01%
Volatility	7.07%	10.98%	2.60%
Correlation with Portfolio	17.41%	89.45%	24.44%

[2] It can be shown that for optimal portfolios, i.e. portfolios with maximum Sharpe Ratio, the component Sharpe Ratios of all assets are equal to the portfolio Sharpe Ration. Therefore, the dispersion of component Sharpe Ratios can be interpretet as deviations from optimality in an ex post analysis.

Given these figures, we can calculate the portfolio Sharpe Ratio. Additionally, we also introduce a benchmark with given expected excess return, volatility and correlation with the portfolio. We will use this information later when attributing the Sharpe Ratio difference.

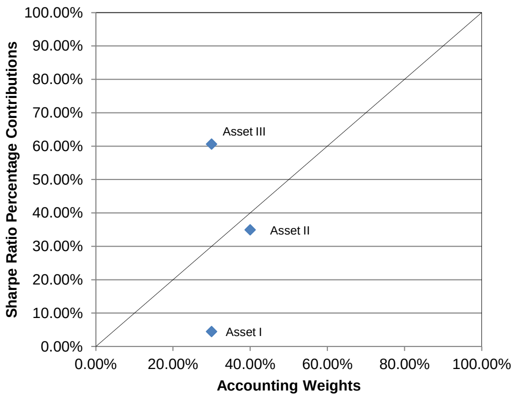
	Portfolio	Benchmark
Expected Return	1.00%	0.96%
Volatility	4.49%	4.10%
Correlation Benchmark	98.98%	100.00%
Sharpe Ratio	0.2219	0.2341
Sharpe Ratio Difference	**-0.0122**	

In order to understand which assets are driving the portfolio's Sharpe Ratio, we apply the decomposition derived above…

	Assets			
	I	II	III	Total
Asset Sharpe Ratio	0.0209	0.0792	0.7752	
Diversification	5.7436	1.1179	4.0914	
Component Sharpe Ratio	**0.1203**	**0.0886**	**3.1715**	
Risk Weight	**8.22%**	**87.54%**	**4.24%**	**100.00%**
Total	**0.0099**	**0.0775**	**0.1345**	**0.2219**

We see that in terms of risk weights, the portfolio is very concentrated, much more than what we would conclude from analyzing accounting exposures (i.e. weights) only. Asset II clearly dominates. On the other hand, asset II has by far the smallest component Sharpe Ratio, resulting in a rather modest contribution to portfolio Sharpe Ratio. It turns out that despite its small risk and accounting weights, asset III delivers the largest contribution to Sharpe Ratio. This is due to its high asset Sharpe Ratio.

This Sharpe Ratio decomposition is very interesting for investment risk management: generally, investment managers as well as risk managers should be interested in portfolio constituents with risk or risk-adjusted performance contributions which differ from accounting exposures. When converting the absolute asset Sharpe Ratio contributions into percentage figures, the following chart can be produced…



In this chart, assets above the 45 degree line are significant positive contributors; assets below are significant negative contributors. The larger an asset's deviation from the 45 degree lines, the less intuitive its impact on the performance and risk dynamics of a portfolio.

The proposed Sharpe Ratio decomposition is not free of issues. For example, the contributions are not independent. This means that varying an asset's weight will impact the risk weight of the other assets (through σ_p and $\rho_{i,p}$) as well as all component Sharpe ratios (through $r_{i,p}$). This is different from simple return contributions, where varying the weight of one asset does not affect the contribution of the other assets. The reason for this dependence this is the non-linearity of risk.

What we have done is to derive additive Sharpe Ratio contributions by sacrificing the independence of contributions. Given the non-additive nature of risk, this is as close to linearity as one can get. Also remember that such additive contributions can only be derived for risk-adjusted performance measures with linear-homogenous risk measures.

Attribution Analysis of the Sharpe Ratio

The goal of the contribution analysis is identifying Sharpe Ratio drivers. In a Sharpe Ratio attribution, we are interested in relating differences between two Sharpe Ratios to management decisions. What we are interested in is to decompose the following expression…

$$S_P - S_B = \frac{r_P - r_f}{\sigma_P} - \frac{r_B - r_f}{\sigma_B}$$

We now make the assumption that decisions are driven by selection (Alpha) and allocation (Beta) considerations, as they are modeled in the single-index model…

$$r_P - r_f = \alpha_P + \beta_P \cdot \left(r_B - r_f\right)$$

Note that this decision framework is consistent with the portfolio-level risk-adjusted performance we are trying to analyze. Modeling allocation and selection decisions like in the Brinson decomposition would not make sense, since we would ignore all aspects of risk and risk-adjusted performance. If we are not interested in risk-adjusted performance, then there is no need to perform a Sharpe Ratio attribution in the first place.

We now make use of the fact that the portfolio beta can be expressed as a volatility ratio times the correlation between portfolio and benchmark…

$$\beta_P = \rho_{P,B} \cdot \frac{\sigma_P}{\sigma_B}$$

$$r_P - r_f = \alpha_P + \rho_{P,B} \cdot \frac{\sigma_P}{\sigma_B} \cdot \left(r_B - r_f\right) = \alpha_P + \rho_{P,B} \cdot \sigma_P \cdot S_B$$

Dividing the above expression by σ_p and rearranging terms yields…

$$\frac{r_P - r_f}{\sigma_P} = \frac{\alpha_P}{\sigma_P} + \rho_{P,B} \cdot S_B$$

$$\frac{r_P - r_f}{\sigma_P} - S_B = \frac{\alpha_P}{\sigma_P} + \rho_{P,B} \cdot S_B - S_B$$

We end up with an expression that decomposes the difference in Sharpe Ratios into two additive components, which we call “active return” and “active risk”…

$$S_P - S_B = \frac{\alpha_P}{\sigma_P} + (\rho_{P,B} - 1) \cdot S_B = ActiveReturn + ActiveRisk$$

The active return component is driven by non-zero portfolio Alphas. The active risk component is the result of a correlation coefficient smaller than 1, i.e. imperfect benchmark tracking.

If we have the portfolio’s alpha and beta in addition to the figures above, we can calculate a “top-down” Sharpe ratio attribution…

	Portfolio	Benchmark
Alpha	-0.04%	
Beta	1.08	

	Sharpe Ratio Attribution
Sharpe Ratio Portfolio	0.2219
Sharpe Ratio Benchmark	0.2341
Sharpe Ratio Difference	**-0.0122**
Active Return Effect	-0.0098
Active Risk Effect	-0.0024
Total Effects	**-0.0122**

Both the active return and active risk effects are negative, although the end result is mainly driven by active returns. The interesting question is whether this aggregate result can be further decomposed and explained by active return and risk contributions from portfolio constituents.

The decomposition of the active return effect is straight forward due to the linearity of Alpha contributions…

$$\alpha_P = \sum_{i=1}^{n} w_i \cdot \alpha_i$$

This allows the same decomposition into three multiplicative components as before. Instead of the portfolio excess return to volatiliy ratio, we decompose the portfolio alpha to portfolio volatility ratio…

$$S_P - S_B = \sum_{i=1}^{n} \frac{w_i \cdot \rho_{i,P} \cdot \sigma_i}{\sigma_P} \cdot \frac{1}{\rho_{i,P}} \cdot \frac{\alpha_i}{\sigma_i} + \left(\rho_{P,B} - 1\right) \cdot S_B$$

The decomposition of the active risk component is less obvious, but can be derived from the linear decomposition of the portfolio's correlation as follows…

$$\rho_{P,B} = \beta_P \cdot \frac{\sigma_B}{\sigma_P} = \sum_{i=1}^{n} w_i \cdot \beta_i \cdot \frac{\sigma_B}{\sigma_P} = \sum_{i=1}^{n} w_i \cdot \rho_{i,B} \cdot \frac{\sigma_i}{\sigma_B} \cdot \frac{\sigma_B}{\sigma_P} = \sum_{i=1}^{n} w_i \cdot \rho_{i,B} \cdot \frac{\sigma_i}{\sigma_P}$$

Finally, we arrive at…

$$S_P - S_B = \sum_{i=1}^{n} \frac{w_i \cdot \rho_{i,P} \cdot \sigma_i}{\sigma_P} \cdot \frac{1}{\rho_{i,P}} \cdot \frac{\alpha_i}{\sigma_i} + w_i \cdot \left(\rho_{i,B} \cdot \frac{\sigma_i}{\sigma_P} - 1\right) \cdot S_B = \sum_{i=1}^{n} ActiveReturn_i + ActiveRisk_i$$

In order to conduct an asset-level Sharpe Ratio attribution, we need the following additional data inputs…

	Assets		
	I	**II**	**III**
Correlation with Portfolio	17.41%	89.45%	24.44%
Alphas	-0.37%	-1.25%	1.88%
Betas	0.54	2.20	0.14

Applying the derived attribution, we get[3]…

Asset-Level Sharpe Ratio Attribution

	Assets			
	I	II	III	**Total**
Active Return Effect	-0.0245	-0.1111	0.1257	**-0.0098**
Active Risk Effect	-0.0359	0.0950	-0.0615	**-0.0024**
Total	**-0.0603**	**-0.0161**	**0.0643**	**-0.0122**

[3] We do not report the decomposition of the active return effects into the three multiplicative components risk weight, diversification and volatility-adjusted asset alphas.

We see in the above table that the main Sharpe Ratio distractor is asset II. In fact, it's the active return component of asset II, due to its negative alpha.

Two interesting observations…

1. Unlike in the Sharpe Ratio contribution analysis above, but like in classical performance attribution analysis, the effects sum vertically as well as horizontally. This is the result of introducing the single-index model.

2. Methods in performance analysis are classified as "external" or "internal", depending on whether or not information about portfolio constituents is available. An interesting aspect of our Sharpe Ratio attribution is the fact that it is "semi-internal" (or "semi-external"): only information about portfolio constituents is required, but not for benchmark constituent information.

Summary and Outlook

We consider it essential for investment managers, risk managers and investors to understand contributions to risk-adjusted performance and attribute differences in risk-adjusted performance, between portfolios to investment decisions on the level of meaningful portfolio segments. Recent advances in risk-adjusted performance analysis have made Sharpe Ratio contribution and attribution possible, filling an empty slot in the toolkit of portfolio analysts.

The decompositions discussed in this paper are ex ante, i.e. they assume constant weights. Ex post versions, taking account changing weights over time, can be derived easily.

It is also straight-forward to generalize the Sharpe Ratio attribution methodology to more relevant multi-factor models. Factor models seem to solve the non-linearity issues in decomposing risk. We have used a factor model in the attribution approach only. Using single or multi factor models in contribution analysis might eliminate the non-additivity and dependence of components across assets.